Date: 30/10/2022

Number of Questions: 24

Time: 3 Hours

Max Marks: 100



Corporate Office : 7-8, Samta Nagar, Hiran Magri, Sector - 3, Udaipur (Rajasthan) - 313001

Answers & Solutions

for IOQM – 2022-23

INSTRUCTIONS TO CANDIDATES

- 1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED**. Only ordinary pens and pencils are allowed inside the examination hall.
- 2. The correction is done by machines through scanning. On the OMR sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use a pencil or a gel pen**. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.
- 3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your score.
- 4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.
- 5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.



- 6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.
- 7. Questions 1 to 10 carry 2 marks each; questions 11 to 22 carry 5 marks each; questions 23 & 24 carry 10 marks each.
- 8. All questions are compulsory.
- 9. There are no negative marks.
- 10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.
- 11. After the exam, you may take away the Candidate's copy of the OMR sheet.
- 12. Preserve your copy of OMR sheet till the end of current Olympiad season. You will need it later for verification purposes.
- 13. You may take away the question paper after the examination.



1. A triangle *ABC* with *AC* = 20 is inscribed in a circle ω . A tangent *t* to ω is drawn through *B*. The distance of *t* from *A* is 25 and that from *C* is 16. If *S* denotes the area of the triangle *ABC*, find the largest integer not exceeding *S*/20.

Answer (10)



2. In a parallelogram *ABCD*, the point *P* on a segment *AB* is taken such that $\frac{AP}{AB} = \frac{61}{2022}$ and a point *Q* on the segment *AD* is taken such that $\frac{AQ}{AD} = \frac{61}{2065}$. If *PQ* intersects *AC* at *T*, find $\frac{AC}{AT}$ to the nearest integer.

Answer (67)





Clearly,
$$\frac{AC}{AT} = \frac{1+\lambda}{1} = \frac{\alpha+\beta}{\alpha\beta} = \frac{1}{\alpha} + \frac{1}{\beta}$$
$$= \frac{2022}{61} + \frac{2065}{61}$$
$$= \frac{4087}{61} = 67$$

3. In a trapezoid ABCD, the internal bisector of angle A intersects the base BC (or its extension) at the point E. Inscribed in the triangle ABE is a circle touching the side AB at M and side BE at the point P. Find the angle DAE in degrees, if AB : MP = 2.

Answer (60)



$$\therefore \quad \frac{x+y}{2} = y \implies x = y$$
$$\therefore \quad \angle B = 60^\circ \implies \angle A = 60^\circ$$

4. Starting with a positive integer *M* written on the board, Alice plays the following game: in each move, if *x* is the number on the board, she replaces it with 3*x* + 2. Similarly, starting with a positive integer *N* written on the board, Bob plays the following game: in each move, if *x* is the number on the board, he replaces it with 2*x* + 27. Given that Alice and Bob reach the same number after playing 4 moves each, find the smallest value of *M* + *N*.

Answer (10)

Sol. In 4 steps Alice will write

 $(3(3(3(3M+2)+2)+2)+2) = \alpha$

and Bob will write

 $(2(2(2(2N+27)+27)+27)+27) = \beta$

As $\alpha = \beta$, we get

81*M* = 16*N* + 325

 \therefore $M_{\min} = 5$ and $N_{\min} = 5$

 $(M + N)_{\min} = 10$



5. Let *m* be the smallest positive integer such that $m^2 + (m + 1)^2 + ... + (m + 10)^2$ is the square of a positive integer *n*. Find *m* + *n*.

Answer (95)

Sol. $\sum_{r=1}^{11} (m+r-1)^2 = 11m^2 + (1^2 + 2^2 + 3^2 + ... + 10^2) + 2m(1+2+3+...+10)$ = 11(m² + 10m + 35) = 11((m + 5)² + 10) The least possible value of m = 18 The required sum = 11(23² + 10) = 11 × 11 × 49 = 77² Then, m = 18 and n = 77 ∴ m + n = 95

6. Let *a*, *b* be positive integers satisfying $a^3 - b^3 - ab = 25$. Find the largest possible value of $a^2 + b^3$.

Answer (43)

Sol. $a^3 - b^3 - ab = 25$ for a = 4 and b = 3

Because for any greater number $a^3 - b^3 - ab > 25$

To prove this if a > b, then $a^3 - b^3 - ab$

 $= (b + t)^3 - b^3 - b(b + t), t > 0$

= $(3t - 1)b^2 + (3t^2 - t)b + t^3$ is always greater than 4,

then
$$b \ge 3$$

So,
$$a^2 + b^3 = 4^2 + 3^3 = 43$$

7. Find the number of ordered pairs (a, b) such that $a, b \in \{10, 11, \dots, 29, 30\}$ and GCD(a, b) + LCM(a, b) = a + b.

Answer (35)

Sol.
$$g + l = a + b$$

 $g + \frac{ab}{g} = a + b$
 $g^2 + (a + b)g + ab = 0$
 $(g - a)(g - b) = 0$
 $g = a, b$
For $a = b$, there will be 21 cases.
If $a = 10, b$ may be 20 or 30 as w
 $a = 11, b = 22$
 $a = 12, b = 24$
 $a = 13, b = 26$
 $a = 14, b = 28$
 $a = 15, b = 30$
Total 35 ways

as well and vice-versa.



8. Suppose the prime numbers *p* and *q* satisfy $q^2 + 3p = 197p^2 + q$. Write $\frac{q}{p}$ as $l + \frac{m}{n}$, where *l*, *m*, *n* are positive integers, *m* < *n* and GCD(*m*, *n*) = 1. Find the maximum value of $l + m + n^2$

integers, m < n and GCD(m, n) = 1. Find the maximum value

Answer (32)

- **Sol.** $q^2 + 3p = 197p^2 + q$
 - $\Rightarrow \quad 197p^2 = q(q-1) + 3p$
 - So, $q 1 = \lambda p$
 - $\therefore \quad 197p^2 = (\lambda p + 1)\lambda p + 3p$
 - \Rightarrow 197 $p = \lambda^2 p + \lambda + 3$
 - $\therefore \quad p = \frac{\lambda + 3}{197 \lambda^2}$ $\therefore \quad \lambda = 14, p = 17$ $\Rightarrow \quad q = 17 \times 14 + 1 = 239$ So, $\frac{q}{p} = 14 + \frac{1}{17}$ $\therefore \quad l + m + n = 32$
- 9. Two sides of an integer sided triangle have lengths 18 and *x*. If there are exactly 35 possible integer values *y* such that 18, *x*, *y* are the side of a non-degenerate triangle, find the number of possible integer values *x* can have.

Answer (Bonus)

Sol. *x* + *y* > 18



So, *x* can take any integer value greater than or equal to 18.

So, answer should be infinite.



10. Consider the 10-digit number M = 9876543210. We obtain a new 10-digit number from M according to the following rule: we can choose one or more disjoint pairs of adjacent digits in M and interchange the digits in these chosen pairs, keeping the remaining digits in their own places. For example, from M = 9876543210, by interchanging the 2 underlined pairs, and keeping the others in their places, we get $M_1 = 9786453210$. Note that any number of (disjoint) pairs can be interchanged. Find the number of new numbers that can be so obtained from M.

Answer (88)

Sol. Number of ways if single pair is changed = $9 = {}^{9}C_{1}$

Number of ways if 2 pairs changed = 7 + 6 + 5 + ... + 1

$$= 28 = {}^{8}C_{2}$$

Number of ways if 3 pairs changed = ${}^{7}C_{3}$

So, total numbers that can be formed = ${}^{9}C_{1} + {}^{8}C_{2} + {}^{7}C_{3} + {}^{6}C_{4} + {}^{5}C_{5}$

= 88

11. Let *AB* be a diameter of a circle ω and let *C* be a point on ω , different from *A* and *B*. The perpendicular from *C* intersects *AB* at *D* and ω at $E(\neq C)$. The circle with centre at *C* and radius *CD* intersects ω at *P* and *Q*. If the perimeter of the triangle *PEQ* is 24, find the length of the side *PQ*.

Answer (08)

Sol.



CPEQ is cyclic quadrilateral $CP \times EQ + CQ \times PE = CE \times PQ$ r(EQ + PE) = 2r PQ 2PQ = EQ + PE PQ + EQ + PE = 24 PQ + 2PQ = 24∴ PQ = 8

12. Given $\triangle ABC$ with $\angle B = 60^{\circ}$ and $\angle C = 30^{\circ}$, let *P*, *Q*, *R* be points on sides *BA*, *AC*, *CB* respectively such that *BPQR* is an isosceles trapezium with *PQ*||*BR* and *BP* = *QR*. Find the maximum possible value of $\frac{2[ABC]}{[BPQR]}$

where [S] denotes the area of any polygon S.

Answer (Bonus)

Sol. *BP* = *QR* = *CR* = *x* & let *BC* = ℓ

 $BR = \ell - x$



 $PQ = (\ell - x) - 2x \cos 60^{\circ}$



Let $f(y) = \frac{y^2}{2y-3}$, f'(y) = 0 gives y = 3 for minimum value.

Minimum value of the expression

$$2\frac{\left[ABC\right]}{\left[BPQR\right]} = \frac{3^2}{2.3 - 3} = 3$$

Maximum value tends to infinity.

13. Let *ABC* be a triangle let *D* be a point on the segment *BC* such that AD = BC. Suppose $\angle CAD = x^\circ$, $\angle ABC = y^\circ$ and $\angle ACB = z^\circ$ and *x*, *y*, *z* are in an arithmetic progression in that order where the first term and the common difference are positive integers. Find the largest possible value of $\angle ABC$ in degrees.

Answer (59)





$$\frac{\sin(x + y + z)}{\sin y} + \frac{\sin x}{\sin z} = 1$$
$$\frac{\sin(3y)}{\sin y} + \frac{\sin(y - d)}{\sin(y + d)} = 1$$
$$\frac{\sin(y - d)}{\sin(y + d)} = -2\cos 2y$$
$$\sin(3y + d) = 0 \implies 3y + d = 180^\circ, 360^\circ$$
$$y \& d \text{ are integers} \implies d \text{ is multiple of } 3$$
$$\text{If } 3y + d = 180^\circ \implies y_{\text{max}} = 59^\circ$$

 $3y + d = 360^{\circ} \Rightarrow y_{max} = 119^{\circ}$ (if y is obtuse, then z must be acute)

Only 1^{st} case is possible $y_{max} = 59^{\circ}$

14. Let *x*, *y*, *z* be complex numbers such that

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = 9$$
$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} = 64$$
$$\frac{x^3}{y+z} + \frac{y^3}{z+x} + \frac{z^3}{x+y} = 488$$

If $\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy} = \frac{m}{n}$ where *m*, *n* are positive integers with GCD (*m*, *n*) = 1, find *m* + *n*.

Answer (16)

Sol.
$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = 9$$

 $(x+y+z)\left(\frac{1}{y+z} + \frac{1}{z+x} + \frac{1}{x+y}\right) = 12$
Let $x + y + z = S_1$
 $\frac{1}{y+z} + \frac{1}{z+x} + \frac{1}{x+y} = \frac{12}{S_1}$
Now, $(x+y+z)\left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right) = 9S_1$
 $64 + S_1 = 9S_1$
 $\Rightarrow S_1 = 8$
 $(x+y+z)\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) = 64S_1$
 $488 + (x^2 + y^2 + z^2) = 64 \times 8$
 $x^2 + y^2 + z^2 = 24$
 $xy + yz + zx = \frac{64-24}{2} = 20$



Now,
$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} = \frac{12}{8} = \frac{3}{2}$$

 $\frac{1}{8-x} + \frac{1}{8-y} + \frac{1}{8-z} = \frac{3}{2}$
 $(8-x)(8-y)(8-z) = 56$
 $\Rightarrow xyz = 104$
So, $\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy} = \frac{24}{104} = \frac{3}{13} = \frac{m}{n}$
 $\Rightarrow m+n = 16$

15. Let x, y be real numbers such that xy = 1. Let T and t be the largest and the smallest values of the expression

$$\frac{\left(x+y\right)^2-\left(x-y\right)-2}{\left(x+y\right)^2+\left(x-y\right)-2}\,.$$

If *T* + *t* can be expressed in the form $\frac{m}{n}$ where *m*, *n* are nonzero integers with GCD (*m*, *n*) = 1, find the value of m + n.

Answer (25)

Sol.
$$xy = 1 \Rightarrow (x + y)^2 = (x - y)^2 + 4$$

$$\frac{(x + y)^2 - (x - y) - 2}{(x + y)^2 + (x - y) - 2} = \frac{(x - y)^2 - (x - y) + 2}{(x - y)^2 + (x - y) + 2}$$

$$= \frac{\left(x - \frac{1}{x}\right)^2 - \left(x - \frac{1}{x}\right) + 2}{\left(x - \frac{1}{x}\right)^2 + \left(x - \frac{1}{x}\right) + 2}$$

$$x - \frac{1}{x} \in \mathbb{R}$$
Let $f(t) = \frac{t^2 - t + 2}{t^2 + t + 2}, t \in \mathbb{R}$

$$\int \frac{f(t)}{f(t)_{max}}$$

$$f(t)_{max} + f(t)_{min} = \frac{4 + \sqrt{2}}{4 - \sqrt{2}} + \frac{4 - \sqrt{2}}{4 + \sqrt{2}} = \frac{18}{7} = \frac{m}{n}$$

$$m + n = 25$$

- 9 -

-

t



16. Let a, b, c be reals satisfying

3ab + 2 = 6b, 3bc + 2 = 5c, 3ca + 2 = 4a.

Let Q denote the set of all rational numbers. Given that the product *abc* can take two values $\frac{r}{s} \in Q$ and $\frac{t}{u} \in Q$,

in lowest form, find r + s + t + u.

Answer (18)

Sol. $3ab + 2 = 6b \dots(i)$, $3bc + 2 = 5c \dots(ii)$, $3ac + 2 = 4a \dots(iii)$ $3abc + 2c = 6bc \dots(1)$ $3abc + 2a = 5ac \dots(2)$

3abc + 2b = 4ab ...(3)

Substitute *bc* from (ii) in (1),

$$3abc = 8c - 4 \qquad \dots (4)$$

Substitute ac from (iii) in (2),

 $3abc + 2a = 5\left(\frac{4a-2}{3}\right)$ 9abc + 6a = 20a - 10 $\boxed{9abc = 14a - 10} \qquad \dots (5)$

Substitute *ab* from (i) in (3),

 $3abc + 2b = 4\left[\frac{6b-2}{3}\right]$ 9bac + 6b = 24b - 8 $\boxed{9abc = 18b - 8} \qquad \dots (6)$

∴ From (4), (5), (6),

 $24c - 12 = 14a - 10 = 18b - 8 = \lambda$

$$c = \frac{12+\lambda}{24}$$
 $a = \frac{10+\lambda}{14}$ $b = \frac{8+\lambda}{18}$

Substitute in (i),

...

$$\mathscr{J}\left(\frac{10+\lambda}{14}\right)\left(\frac{8+\lambda}{1\cancel{8}6}\right)+2=\mathscr{J}\left(\frac{8+\lambda}{1\cancel{8}3}\right)$$
$$(\lambda+8)(\lambda+10)+84\times2=28(8+\lambda)$$
$$\lambda^{2}+18\lambda+80+168=224+28\lambda$$
$$\lambda^{2}-10\lambda+24=0$$
$$(\lambda-6)(\lambda-4)=0$$
$$\lambda=6,4$$
For $\lambda=6$

$$a = \frac{16}{14}$$
 $b = \frac{14}{18}$ $c = \frac{18}{24}$



$$\therefore \quad abc = \frac{16}{14} \times \frac{14}{18} \times \frac{18}{24} = \frac{16}{24} = \frac{2}{3}$$
For $\lambda = 4$,
 $a = 1 \quad b = \frac{12}{18} \quad c = \frac{16}{24}$
 $abc = \frac{4}{9}$

$$r + s + t + u = 18$$

17. For a positive integer n > 1, let g(n) denote the largest positive proper divisor of n and f(n) = n - g(n). For example, g(10) = 5, f(10) = 5 and g(13) = 1, f(13) = 12. Let N be the smallest positive integer such that f(f(f(N))) = 97. Find the largest integer not exceeding \sqrt{N} .

Answer (19)

Sol. If f(n) = x and x is a prime then least value of n = 2x ...(1)

and if f(n) = x and x is composite but x + 1 is a prime then least value of n = x + 1 ...(2)

 $\therefore \quad f(f(f(n))) = 97$

Then, f(f(n)) = 194 [from (1)]

Now, f(n) for *n* to be least can be 3×97 or 4×97

Case I: $f(n) = 3 \times 97$, then least value of $n = 6 \times 97$

Case II : $f(n) = 4 \times 97$, then least value of $n = 4 \times 97 + 1$ from equation (2)

The smallest positive value of $n = 4 \times 97 + 1$

∴ *N* = 389

$$\therefore \quad \sqrt{N} = \sqrt{389} > 19$$

18. Let *m*, *n* be natural numbers such that

m + 3n - 5 = 5LCM(m, n) - 11GCD(m, n).

Find the maximum possible value of m + n.

Answer (70)

Sol. Let G. C. D. of (*m*, *n*) = *d*

Then for some positive coprime integers *x* and *y*

```
m = dx and n = dy
```

- :: m + 3n 5 = 2 LCM(m, n) 11 GCD(m, n)
- $\therefore \quad dx + 3dy 5 = 2dxy 11d$
- or, d(x + 3y 2xy + 11) = 5

Now, to maximize the sum m + n, d must be 5

- \therefore x + 3y 2xy + 11 = 1
- or x + 3y 2xy + 10 = 0
- or (2x-3)(2y-1) = 23



Case I: 2x - 3 = 1 and 2y - 1 = 23

 \therefore x = 2 and y = 12

This is not possible as x, y are coprime

Case II : 2*x* – 3 = 23 and 2*y* – 1 = 1

- \therefore x = 13 and y = 1
- $\therefore m + n = (13 + 1) \times 5 = 70$
- 19. Consider a string of *n* 1's. We wish to place some + signs in between so that the sum is 1000. For instance, if n = 190, one may put + signs so as to get 11 ninety times and 1 ten times, and get the sum 1000. If *a* is the number of positive integers *n* for which it is possible to place + signs so as to get the sum 1000, then find the sum of the digits of *a*.

Answer (09)

Sol. :: $1000 = 1 \cdot a_1 + 11 \cdot a_2 + 111 \cdot a_3 + \dots$

where a_1 , a_2 , a_3 , are non-negative integers for all a_i , when i > 3, $a_i = 0$

 \therefore 1000 = 111*p* + 11*q* + *r*

If p = 0, then there are 91 possibilities for q, r.

For p = 1, there are 81 possibilities for q, r.

For p = 2, there are 71 possibilities for q, r.

For p = 3, there are 61 possibilities for q, r.

For p = 4, there are 51 possibilities for q, r.

.....

.....

.....

For p = 9, there are 1 possibility for q, r.

For p = 0 and p = 1, only two new values of 'a' are generated.

Similarly, for p = 1 and p = 2, only two new values of 'a' are generated and so on up to p = 8.

 \therefore Total number of possibilities = 91 + 2 × 8 + 1 = 108

Hence, sum of digits of a = 9

Note : Example for n = 892

(a) If p = 0, q = 12 and r = 868,

1000 = 111(0) + 11(12) + (868)

```
(b) If p = 1, q = 0, r = 889,
```

1000 = 111(1) + 11(0) + (889)

- ... All such possibilities are to be counted once.
- 20. For an integer $n \ge 3$ and a permutation $\sigma = (p_1, p_2, ..., p_n)$ of $\{1, 2, ..., n\}$, we say p_1 is a *landmark point* if $2 \le l \le n 1$ and $(p_{l-1} p_l)(p_{l+1} p_l) > 0$. For example, for n = 7, the permutation (2, 7, 6, 4, 5, 1, 3) has four landmark points: $p_2 = 7$, $p_4 = 4$, $p_5 = 5$ and $p_6 = 1$. For a given $n \ge 3$, let L(n) denote the number of permutation of $\{1, 2, ..., n\}$ with exactly only landmark point. Find the maximum $n \le 3$ for which L(n) is a perfect square.

Answer (03)



Sol. For the permutations of set {1, 2, 3, ..., *n*}, the landmark point should be 1 or *n* to satisfy given conditions.

If *n* is at (similarly for 1) *r*th position, there is only one permutation of the remaining numbers for each selection.

So, number of selections =
$$\sum_{r=2}^{n-1} {}^{n-1}C_{r-1}$$
$$= 2^{n-1} - 2$$

 $\therefore \quad \text{Total number of selections} = 2(2^{n-1}-2)$

$$L(n) = 4(2^{n-2} - 1)$$

Now for L(n) to be a perfect square *n* should be equal to 3.

21. An ant is at a vertex of a cube. Every 10 minutes it moves to an adjacent vertex along an edge. If *N* is the number of one hour journeys that end at the starting vertex, find the sum of the squares of the digits of *N*.

Answer (74)

Sol. We have divided vertices into four categories



 $X \rightarrow$ Starting vertex

 $Y \rightarrow Adjacent vertex$

 $Z \rightarrow Adjacent to Y but not same as X$

 $W \rightarrow Adjacent to Z but not same as Y$

Let a_n = number of ways that after n steps ant is at X

 b_n = number of ways that after *n* steps ant is at Y

 c_n = number of ways that after *n* steps ant is at *Z*

 d_n = number of ways that after *n* steps ant is at *W*

We need to find a6

	$a_{n+1} = 3b_n$	(i)
	$b_{n+1} = a_n + 2c_n$	(ii)
	$c_{n+1} = 2b_n + d_n$	(iii)
and	$d_{n+1} = 3c_n$	(iv)

By eliminating b_n , c_n and d_n we get

 $a_{n+3} = 10a_{n+1} - 9a_{n-1}$



- \therefore $a_1 = 0, a_2 = 3, a_3 = 0$ and $a_4 = 21$
- $\therefore \quad a_6 = 10a_4 9a_2 = 210 27 = 183$
- ∴ *N* = 183
- Sum of square of digits of N = 74
- 22. A binary sequence is a sequence in which each term is equal to 0 or 1. A binary sequence is called friendly if each term is adjacent to at least one term that is equal to 1. For example, the sequence 0, 1, 1, 0, 0, 1, 1, 1 is friendly. Let F_n denote the number of friendly binary sequences with *n* terms. Find the smallest positive integer $n \ge 2$ such that $F_n > 100$.

Answer (11)

- **Sol.** Let a_n = number of friendly sequences ending with 0
 - B_n = number of friendly sequences ending with 1

 $F_n = a_n + b_n$...(i)p Now, $a_{n+1} = b_n$...(ii) (by adding 0 in the last) and $b_{n+1} = a_n + b_n + a_{n-1}$...(iii) \therefore $F_n = a_n + a_{n+1}$ from equation (i) and equation (ii) and $a_{n+2} = a_{n+1} + a_n + a_{n-1}$ by (ii) and (iii) \therefore $a_1 = 0, a_2 = 1, a_3 = 3$ So, $a_4 = 4, a_5 = 5, a_6 = 9, a_7 = 16, a_8 = 25, a_9 = 39, a_{10} = 64$ and $a_{11} = 105$ and so on. Clearly, we can see that $F_{11} = 105$.

So, $F_{11} > 100$.

In a triangle ABC, the median AD divides ∠BAC in the ratio 1 : 2. Extend AD to E such that EB is perpendicular AB. Given that BE = 3, BA = 4, find the integer nearest to BC².

Answer (29)

Sol.



Here, *D* is mid-point of *BC*, hence *BD* : *CD* = 1 : 1

Let $\angle BAD = \theta$, then $\angle CAD = 2\theta$

 \therefore tan $\theta = \frac{3}{4}$, and tan $2\theta = \frac{24}{7}$



Now, using $\cot m - n$ theorem in $\triangle ABC$

 $2\cot\alpha = \cot\theta - \cot2\theta$

$$\Rightarrow \quad \cot \alpha = \frac{25}{48}$$

Now, using sine rule in $\triangle ABD$, we get

$$\frac{BD}{4} = \frac{\sin\theta}{\sin(\pi - \alpha)}$$
$$\therefore \qquad BD = \frac{4 \times 3\sqrt{25^2 + 48^2}}{5 \times 48}$$

So,
$$4BD^2 = BC^2 = \frac{25^2 + 48^2}{100} = 29.29$$

Nearest integer is 29.

24. Let *N* be the number of ways of distributing 52 identical balls into 4 distinguishable boxes such that no box is empty and the difference between the number of balls in any two of the boxes is not a multiple of 6. If N = 100a + b, where *a*, *b* are positive integers less than 100, find a + b.

Answer (81)

Sol. Let i^{th} box has $6\lambda_i + \mu_i$ balls where $\lambda_i, \mu_i \in w$ and $\mu_i \leq 5$. Also, all μ_i 's are distinct.

 $\therefore \quad 6(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) + (\mu_1 + \mu_2 + \mu_3 + \mu_4) = 52 \text{ and } \mu_1 + \mu_2 + \mu_3 + \mu_4 \in \{6, 7, 8, 9, 10, 11, 12, 13, 14\}$

Hence, only one possibility is there

 $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 7 \qquad \dots (i)$

and $\mu_1 + \mu_2 + \mu_3 + \mu_4 = 10$...(ii)

Equation (ii) has only three solution sets, which are (5, 4, 1, 0), (5, 3, 2, 0) and (4, 3, 2, 1).

Equation (i) has total ${}^{10}C_3$ solutions and ${}^{4}C_1 \cdot {}^{9}C_2$ solutions when any λ_i is zero.

So,
$$N = 3({}^{10}C_3 \cdot (4!)) - 2({}^{4}C_1 \cdot {}^{9}C_2(3!)) = 6912$$

∴ *a* = 69 and *b* = 12.

