16. Let a, b, c be reals satisfying

$$3ab + 2 = 6b,\ 3bc + 2 = 5c,\ 3ca + 2 = 4a.$$

Let Q denote the set of all rational numbers. Given that the product abc can take two values $\frac{r}{s} \in Q$ and $\frac{t}{u} \in Q$, in lowest form, find $r + s + t + u$.

Answer (18)

Sol. $3ab + 2 = 6b$ …(i), $3bc + 2 = 5c$ …(ii), $3ac + 2 = 4a$ …(iii)

$3abc + 2c = 6bc$ …(1)

$3abc + 2a = 5ac$ …(2)

$3abc + 2b = 4ab$ …(3)

Substitute bc from (ii) in (1),

$$3abc + 2c = 10c - 4$$

$$\boxed{3abc = 8c - 4} \quad \text{…(4)}$$

Substitute ac from (iii) in (2),

$$3abc + 2a = 5\left(\frac{4a-2}{3}\right)$$

$$9abc + 6a = 20a - 10$$

$$\boxed{9abc = 14a - 10} \quad \text{…(5)}$$

Substitute ab from (i) in (3),

$$3abc + 2b = 4\left[\frac{6b-2}{3}\right]$$

$$9bac + 6b = 24b - 8$$

$$\boxed{9abc = 18b - 8} \quad \text{…(6)}$$

$\therefore$ From (4), (5), (6),

$$24c - 12 = 14a - 10 = 18b - 8 = \lambda$$

$$c = \frac{12+\lambda}{24} \quad a = \frac{10+\lambda}{14} \quad b = \frac{8+\lambda}{18}$$

Substitute in (i),

$$\cancel{3}\left(\frac{10+\lambda}{14}\right)\left(\frac{8+\lambda}{\cancel{18}6}\right) + 2 = \cancel{6}\left(\frac{8+\lambda}{\cancel{18}3}\right)$$

$$(\lambda + 8)(\lambda + 10) + 84 \times 2 = 28(8 + \lambda)$$

$$\lambda^2 + 18\lambda + 80 + 168 = 224 + 28\lambda$$

$$\lambda^2 - 10\lambda + 24 = 0$$

$$(\lambda - 6)(\lambda - 4) = 0$$

$$\lambda = 6, 4$$

$\therefore$ For $\lambda = 6$

$$a = \frac{16}{14} \quad b = \frac{14}{18} \quad c = \frac{18}{24}$$

$\therefore \quad abc = \frac{16}{14} \times \frac{14}{18} \times \frac{18}{24} = \frac{16}{24} = \frac{2}{3}$

For $\lambda = 4$,

$a = 1 \quad b = \frac{12}{18} \quad c = \frac{16}{24}$

$abc = \frac{4}{9}$

$r + s + t + u = 18$

17. For a positive integer $n > 1$, let $g(n)$ denote the largest positive proper divisor of n and $f(n) = n - g(n)$. For example, $g(10) = 5$, $f(10) = 5$ and $g(13) = 1$, $f(13) = 12$. Let N be the smallest positive integer such that $f(f(f(N))) = 97$. Find the largest integer not exceeding $\sqrt{N}$.

Answer (19)

Sol. If $f(n) = x$ and x is a prime then least value of $n = 2x$ …(1)

and if $f(n) = x$ and x is composite but $x + 1$ is a prime then least value of $n = x + 1$ …(2)

$\because \quad f(f(f(n))) = 97$

Then, $f(f(n)) = 194$ [from (1)]

Now, $f(n)$ for n to be least can be 3×97 or 4×97

Case I : $f(n) = 3 \times 97$, then least value of $n = 6 \times 97$

Case II : $f(n) = 4 \times 97$, then least value of $n = 4 \times 97 + 1$ from equation (2)

The smallest positive value of $n = 4 \times 97 + 1$

$\therefore \quad N = 389$

$\therefore \quad \sqrt{N} = \sqrt{389} > 19$

18. Let m, n be natural numbers such that

$$m + 3n - 5 = 5\text{LCM}(m, n) - 11\text{GCD}(m, n).$$

Find the maximum possible value of $m + n$.

Answer (70)

Sol. Let G. C. D. of $(m, n) = d$

Then for some positive coprime integers x and y

$m = dx$ and $n = dy$

$\because \quad m + 3n - 5 = 2\text{ LCM }(m, n) - 11\text{ GCD }(m, n)$

$\therefore \quad dx + 3dy - 5 = 2dxy - 11d$

or, $\quad d(x + 3y - 2xy + 11) = 5$

Now, to maximize the sum $m + n$, d must be 5

$\therefore \quad x + 3y - 2xy + 11 = 1$

or $\quad x + 3y - 2xy + 10 = 0$

or $\quad (2x - 3)(2y - 1) = 23$

Case I : $2x - 3 = 1$ and $2y - 1 = 23$

$\therefore$ $x = 2$ and $y = 12$

This is not possible as x, y are coprime

Case II : $2x - 3 = 23$ and $2y - 1 = 1$

$\therefore$ $x = 13$ and $y = 1$

$\therefore$ $m + n = (13 + 1) \times 5 = 70$

19. Consider a string of n 1's. We wish to place some + signs in between so that the sum is 1000. For instance, if $n = 190$, one may put + signs so as to get 11 ninety times and 1 ten times, and get the sum 1000. If a is the number of positive integers n for which it is possible to place + signs so as to get the sum 1000, then find the sum of the digits of a.

Answer (09)

Sol. $\because$ $1000 = 1 \cdot a_1 + 11 \cdot a_2 + 111 \cdot a_3 + \ldots\ldots$

where $a_1, a_2, a_3, \ldots\ldots$ are non-negative integers for all a_i, when $i > 3$, $a_i = 0$

$\therefore$ $1000 = 111p + 11q + r$

If $p = 0$, then there are 91 possibilities for q, r.

For $p = 1$, there are 81 possibilities for q, r.

For $p = 2$, there are 71 possibilities for q, r.

For $p = 3$, there are 61 possibilities for q, r.

For $p = 4$, there are 51 possibilities for q, r.

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For $p = 9$, there are 1 possibility for q, r.

For $p = 0$ and $p = 1$, only two new values of 'a' are generated.

Similarly, for p = 1 and p = 2, only two new values of 'a' are generated and so on up to $p = 8$.

$\therefore$ Total number of possibilities $= 91 + 2 \times 8 + 1 = 108$

Hence, sum of digits of $a = 9$

Note : Example for $n = 892$

(a) If $p = 0$, $q = 12$ and $r = 868$,

$$1000 = 111(0) + 11(12) + (868)$$

(b) If $p = 1$, $q = 0$, $r = 889$,

$$1000 = 111(1) + 11(0) + (889)$$

$\therefore$ All such possibilities are to be counted once.

20. For an integer $n \geq 3$ and a permutation $\sigma = (p_1, p_2, \ldots, p_n)$ of $\{1, 2, \ldots, n\}$, we say p_1 is a *landmark point* if $2 \leq l \leq n - 1$ and $(p_{l-1} - p_l)(p_{l+1} - p_l) > 0$. For example, for $n = 7$, the permutation (2, 7, 6, 4, 5, 1, 3) has four landmark points: $p_2 = 7$, $p_4 = 4$, $p_5 = 5$ and $p_6 = 1$. For a given $n \geq 3$, let $L(n)$ denote the number of permutation of $\{1, 2, \ldots, n\}$ with exactly only landmark point. Find the maximum $n \leq 3$ for which $L(n)$ is a perfect square.

Answer (03)

Sol. For the permutations of set {1, 2, 3, …, n}, the landmark point should be 1 or n to satisfy given conditions.

$\underline{\quad}_{I^{st}}\ \underline{\quad}_{II^{nd}}\ \_\ \_\ \underline{\quad}_{r^{th}}\ \_\ \_\ \_\ \underline{\quad}_{n^{th}}$

If n is at (similarly for 1) r^{th} position, there is only one permutation of the remaining numbers for each selection.

So, number of selections = $\sum_{r=2}^{n-1} {}^{n-1}C_{r-1}$

$= 2^{n-1} - 2$

$\therefore$ Total number of selections = $2(2^{n-1} - 2)$

$L(n) = 4(2^{n-2} - 1)$

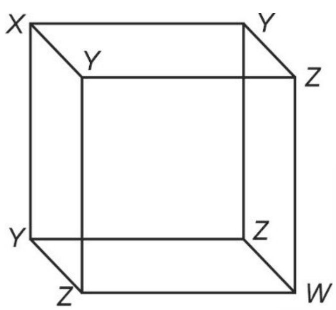
Now for $L(n)$ to be a perfect square n should be equal to 3.

$\therefore$ $n = 3$

21. An ant is at a vertex of a cube. Every 10 minutes it moves to an adjacent vertex along an edge. If N is the number of one hour journeys that end at the starting vertex, find the sum of the squares of the digits of N.

Answer (74)

Sol. We have divided vertices into four categories



$X \rightarrow$ Starting vertex

$Y \rightarrow$ Adjacent vertex

$Z \rightarrow$ Adjacent to Y but not same as X

$W \rightarrow$ Adjacent to Z but not same as Y

Let a_n = number of ways that after n steps ant is at X

b_n = number of ways that after n steps ant is at Y

c_n = number of ways that after n steps ant is at Z

d_n = number of ways that after n steps ant is at W

We need to find a_6

$a_{n+1} = 3b_n$ …(i)

$b_{n+1} = a_n + 2c_n$ …(ii)

$c_{n+1} = 2b_n + d_n$ …(iii)

and $d_{n+1} = 3c_n$ …(iv)

By eliminating b_n, c_n and d_n we get

$a_{n+3} = 10a_{n+1} - 9a_{n-1}$

∵ $a_1 = 0$, $a_2 = 3$, $a_3 = 0$ and $a_4 = 21$

∴ $a_6 = 10a_4 - 9a_2 = 210 - 27 = 183$

∴ $N = 183$

Sum of square of digits of $N = 74$

22. A binary sequence is a sequence in which each term is equal to 0 or 1. A binary sequence is called friendly if each term is adjacent to at least one term that is equal to 1. For example, the sequence 0, 1, 1, 0, 0, 1, 1, 1 is friendly. Let F_n denote the number of friendly binary sequences with n terms. Find the smallest positive integer $n \geq 2$ such that $F_n > 100$.

Answer (11)

Sol. Let a_n = number of friendly sequences ending with 0

B_n = number of friendly sequences ending with 1

$F_n = a_n + b_n$ …(i)p

Now, $a_{n+1} = b_n$ …(ii) (by adding 0 in the last)

and $b_{n+1} = a_n + b_n + a_{n-1}$ …(iii)

∵ $F_n = a_n + a_{n+1}$ from equation (i) and equation (ii)

and $a_{n+2} = a_{n+1} + a_n + a_{n-1}$ by (ii) and (iii)

∵ $a_1 = 0$, $a_2 = 1$, $a_3 = 3$

So, $a_4 = 4$, $a_5 = 5$, $a_6 = 9$, $a_7 = 16$, $a_8 = 25$, $a_9 = 39$, $a_{10} = 64$ and $a_{11} = 105$ and so on.

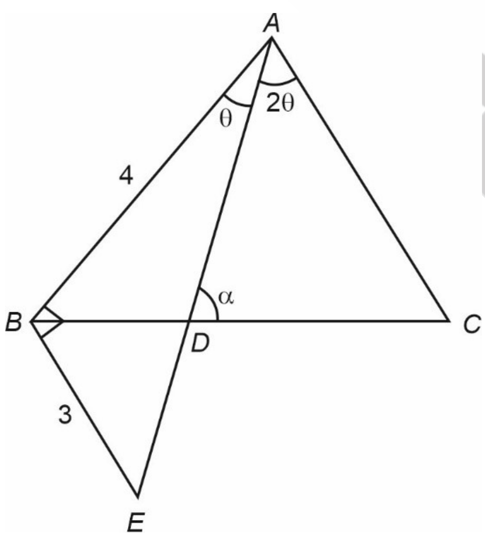
Clearly, we can see that $F_{11} = 105$.

So, $F_{11} > 100$.

23. In a triangle ABC, the median AD divides $\angle BAC$ in the ratio 1 : 2. Extend AD to E such that EB is perpendicular AB. Given that $BE = 3$, $BA = 4$, find the integer nearest to BC^2.

Answer (29)

Sol.



Here, D is mid-point of BC, hence $BD : CD = 1 : 1$

$\angle ABE = 90^\circ$

Let $\angle BAD = \theta$, then $\angle CAD = 2\theta$

∴ $\tan\theta = \frac{3}{4}$, and $\tan 2\theta = \frac{24}{7}$

Now, using cot $m - n$ theorem in ΔABC

$$2\cot\alpha = \cot\theta - \cot 2\theta$$

$$\Rightarrow \quad \cot\alpha = \frac{25}{48}$$

Now, using sine rule in ΔABD, we get

$$\frac{BD}{4} = \frac{\sin\theta}{\sin(\pi - \alpha)}$$

$$\therefore \quad BD = \frac{4 \times 3\sqrt{25^2 + 48^2}}{5 \times 48}$$

So, $4BD^2 = BC^2 = \dfrac{25^2 + 48^2}{100} = 29.29$

Nearest integer is 29.

24. Let N be the number of ways of distributing 52 identical balls into 4 distinguishable boxes such that no box is empty and the difference between the number of balls in any two of the boxes is not a multiple of 6. If $N = 100a + b$, where a, b are positive integers less than 100, find $a + b$.

Answer (81)

Sol. Let i^{th} box has $6\lambda_i + \mu_i$ balls where $\lambda_i, \mu_i \in w$ and $\mu_i \le 5$. Also, all μ_i's are distinct.

$\because \quad 6(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) + (\mu_1 + \mu_2 + \mu_3 + \mu_4) = 52$ and $\mu_1 + \mu_2 + \mu_3 + \mu_4 \in \{6, 7, 8, 9, 10, 11, 12, 13, 14\}$

Hence, only one possibility is there

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 7 \qquad \ldots(i)$

and $\mu_1 + \mu_2 + \mu_3 + \mu_4 = 10 \qquad \ldots(ii)$

Equation (ii) has only three solution sets, which are (5, 4, 1, 0), (5, 3, 2, 0) and (4, 3, 2, 1).

Equation (i) has total ${}^{10}C_3$ solutions and ${}^4C_1 \cdot {}^9C_2$ solutions when any λ_i is zero.

So, $N = 3({}^{10}C_3 \cdot (4!)) - 2\left({}^4C_1 \cdot {}^9C_2 (3!)\right) = 6912$

$\therefore \quad a = 69$ and $b = 12$.

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